Redundancy in the Discrete-Vortex Method for Closed Bodies

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Introduction

It is hoped that this Note will provide some additional insight with regard to the redundancy that occurs in the no-penetration condition when vortex-lattice methods are applied to closed bodies. The problem is discussed in the context of the vortex-lattice method, but the phenomenon also appears elsewhere. The velocity field generated by a closed loop of constant-circulation, discrete-vortex segments along the edges of a planar element is identical to the one generated by a distribution of uniform-strength doublets over the surface of the element.[1] Because vortex-lattice methods do not require the introduction of local (elemental) coordinate systems or approximate planar elements, they are relatively simple and computationally inexpensive; thus, they are an appealing alternative to source-panel methods.

The use of vorticity as the surface-distributed singularity for thick bodies has often been criticized, sources being preferred. But from the continuity equation for incompressible flows and the definition of vorticity, it follows that vorticity anywhere in the flowfield generates velocity everywhere in the flowfield. This purely kinematical result is valid for both viscous as well as inviscid flows. In an actual flow, the velocity generated by the vorticity in the boundary layers and wakes interferes with the oncoming stream to the extent that both the no-slip and no-penetration conditions are satisfied. Thus, vorticity alone should be adequate to model incompressible flows. The adequacy of the method is illustrated with a comparison of computed and exact results.

The vortex-lattice method has been successfully applied to the computation of flows over lifting surfaces and around closed bodies,[2-6] but the points discussed here have not been published previously. Comparisons of computed and experimental results and more detailed explanations of the foundations of vorticity-based methods can be found in Ref. 7.

Vortex-Lattice Method

An arbitrary closed body is discretized (or approximated) by an arbitrary number of quadrilateral elements. The corners of the elements lie on the actual surface and are connected by straight-line segments. Generally, the elements are nonplanar. Each element is encircled by a closed loop of discrete-vortex segments lying along its edges. The result is a net or lattice of discrete-vortex lines that covers the surface. The requirement of spatial conservation of circulation is satisfied by taking the circulations around all the individual vortex segments of a given loop to be the same. For element number i, the value of this loop circulation is G_i. Adjoining loops have one segment in common, and the circulation around that common segment is the difference between the circulations around the two loops. For example, in Fig. 1, the circulation around leg j of the loop that encircles element i is given by

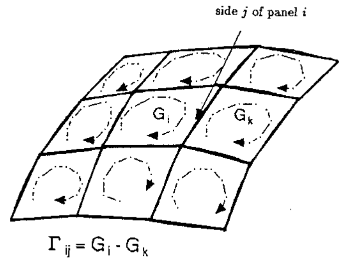

Fig. 1 Vortex lattice.

$$\Gamma_{ij} = G_i - G_k \quad (i \text{ and } k = 1, 2, \ldots, N, \quad j = 1, 2, 3, 4) \tag{1}$$

where N is the number of panels, $k = NB(i, j)$ and NB is the so-called neighbor matrix, whose elements provide the index of the panel that "neighbors" panel i along its side j.

The G_i are determined by imposing the no-penetration condition, which can be expressed as follows:

$$(\boldsymbol{V}_\infty + \boldsymbol{V}_D)\cdot\boldsymbol{n} = 0 \tag{2}$$

where $\boldsymbol{V}_\infty$ is the velocity in the oncoming stream, $\boldsymbol{V}_D$ is the so-called disturbance velocity generated by the body, and $\boldsymbol{n}$ is the unit vector normal to the surface. In the numerical problem, Eq. (2) is satisfied at only one point in each element, the so-called control point, which is the centroid of the corners; $\boldsymbol{n}$ is the cross-product of the two diagonals; and $\boldsymbol{V}_D\cdot\boldsymbol{n}$ is expressed in the terms of the G_i

$$(\boldsymbol{V}_D\cdot\boldsymbol{n})_{\text{at element } i} = \sum_{j=1}^{N} A_{ij}G_j = -\boldsymbol{V}_\infty\cdot\boldsymbol{n}_i \quad (i = 1, \ldots, N) \tag{3}$$

where A_{ij} is the normal component of the velocity generated at the control point of element i by the unit-circulation loop of discrete-vortex segments enclosing element j, and G_j is the actual circulation around loop j. The A_{ij} are computed by repeated application of the Biot-Savart law. The control point may not quite lie on the surface of the panel, and $\boldsymbol{n}$ is an approximate normal. The no-penetration condition can be expressed in the matrix form:

$$[A]\{G\} = \{R\} \tag{4}$$

Explanation for the Singular Matrix

The matrix $[A]$ in Eq. (4) is singular for a closed body, and nonsingular for an open surface. Such behavior can be explained by the continuity equation. For a closed surface with an interior domain, it follows from the continuity requirement that if the no-penetration condition is satisfied on $N - 1$ panels, then the remaining panel is automatically impermeable. Hence, the no-penetration condition for this panel is redundant, system (4) is overdetermined, and $[A]$ is singular.

There is another aspect to the difference between lifting-surface problems and closed-body problems. For lifting surfaces the quadrilaterals along the edges do not have neighbors on all sides. Consequently, the value of the loop circulation is also the value of the circulation around the vortex segment along a portion of the edge of the lifting surface. A vortex along the edge could represent the so-called starting vortex in an unsteady flow. In any case, the actual values of the G_i must be unique. In contrast, for closed bodies, the circulations around every segment are the differences between the loop circulations of the adjoining elements, as explained above. Moreover, the velocity generated by a uniform-circulation

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vortex lattice spread over a closed surface is zero everywhere. Hence, only the differences in the loop circulations must be unique for a closed body and, consequently, there is some arbitrariness associated with their values. The following example will illustrate the latter point.

Illustrative Example

A sphere is discretized by two panels in the axial direction and two panels in the radial direction (symmetry is assumed about XZ plane; see Fig. 2). The flow is in the x direction. For this simple, closed configuration, the coefficients and the right sides in Eq. (4) are as follows:

$$[A] = \begin{vmatrix} -0.8016 & 0.4931 & 0.1742 & 0.1343 \\ 0.4931 & -0.8016 & 0.1343 & 0.1742 \\ 0.1742 & 0.1343 & -0.8016 & 0.4931 \\ 0.1343 & 0.1742 & 0.4931 & -0.8016 \end{vmatrix} \qquad \{R\} = \begin{vmatrix} 0.5774 \\ 0.5774 \\ -0.5774 \\ -0.5774 \end{vmatrix} \tag{5}$$

The determinant of $[A]$ is zero. To solve this overdetermined system of linear equations, one can assume that the first equation in Eqs. (4) and (5), which gives the flow through the first panel, is redundant, as explained in the previous section; one can then assign an arbitrary value to the loop circulation on the first panel G_1 (e.g., $G_1 = 1.0$), thereby reducing the order of the system by one; and finally one can solve the remaining $N - 1$ (in this example 3) equations. Substituting $G_1 = 1$ into the last three equations gives the following:

$$[A] = \begin{vmatrix} -0.8016 & 0.1343 & 0.1742 \\ 0.1343 & -0.8016 & 0.4931 \\ 0.1742 & 0.4931 & -0.8016 \end{vmatrix}, \{R\} = \begin{vmatrix} 0.0843 \\ -0.7515 \\ -0.7117 \end{vmatrix} \tag{6}$$

The solution of the corresponding set of equations is

$$\{G\}^T = [1.0 \quad 1.0 \quad 2.8714 \quad 2.8714] \tag{7}$$

Substituting $\{G\}$ into the redundant first equation shows that it is actually a solution of the original four equations. The corresponding $[\Gamma]$ is computed from Eq. (1):

$$[\Gamma] = \begin{vmatrix} 0 & 0 & -1.8714 & 0 \\ 0 & 0 & -1.8714 & 0 \\ 1.8714 & 0 & 0 & 0 \\ 1.8714 & 0 & 0 & 0 \end{vmatrix}$$

where

$$[NB] = \begin{vmatrix} 2 & 1 & 3 & 2 \\ 1 & 1 & 4 & 2 \\ 1 & 3 & 4 & 4 \\ 2 & 3 & 3 & 4 \end{vmatrix} \tag{8}$$

The same solution for $[\Gamma]$ is obtained for different choices for G_1. If in the procedure above, one chooses G_1 to be zero, the effect is the same as eliminating one element, which is the common practice. A lattice of uniform circulation over a closed body does not generate any velocity anywhere [a manifestation of this fact is that the sums of the elements in the rows of $[A]$ in Eq. (5) are zero]; therefore, there exists an infinite number of solutions to the original problem. Arbitrarily assigning a value to one of the G_i (which has the effect of eliminating a panel) will produce one of the solutions. The solution for $[\Gamma]$ is unique. The results do not depend on which panel is eliminated.

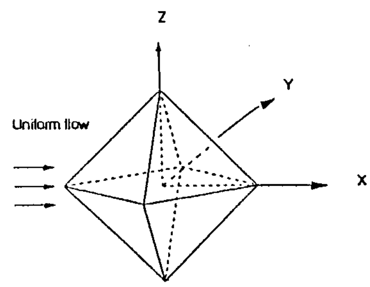

Fig. 2 Four-panel discretization of the sphere.

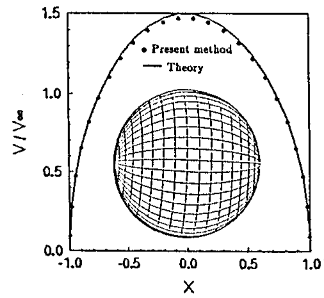

Fig. 3 Comparison of numerical and exact results for a sphere.

Often $[A]$ is diagonally dominant, and the system of all N equations can be solved by iteration. When this happens, there is no need to assign a value to one of the G_i. In spite of the fact that the determinant of $[A]$ is zero, the iteration converges to a correct solution. Different initial guesses lead to different values for the G_i; however, the circulations around the legs are always the same, as they are with the method discussed above.

When the density of the panels is increased to a moderate level, the computed-velocity distribution is in good agreement with the exact solution, as shown in Fig. 3. Results for other bodies are also in good agreement with exact solutions. For the details of the computation of the velocity at the surface, the reader is referred to Srivastava.[5]

Conclusions

The redundancy that occurs when vortex-lattice methods are applied to closed bodies is discussed. The solution to the no-penetration condition in terms of the circulations around the closed loops encircling the elements of the lattice is not unique, but the circulations around the individual vortex segments of the lattice are. Two procedures to solve the no-penetration condition are explained and illustrated by means of a simple example in which the flow over a sphere is modeled with a very crude mesh. Then the mesh is refined, and it is shown that the procedure produces an accurate approximation to the exact solution.

References

[1]Lamb, H., *Hydrodynamics*, Article 145, Dover, New York, 1945.

[2]Margason, R. J., and Lamar, J. E., "Vortex-Lattice FORTRAN Program for Estimating Subsonic, Aerodynamic Characteristics of Complex Planforms," NASA TN-D-6142, 1978.

[3]"Vortex Lattice Utilization," NASA SP-405, May 1980.

[4]Asfar, K. R., Mook, D. T., and Nayfeh, A. H., "Application of the Vortex-Lattice Technique to Arbitrary Bodies," *Journal of Aircraft*, Vol. 16, No. 3, 1979, pp. 421–424.

[5]Srivastava, A., "Quadrilateral Vortex Panel Code—Theory Document," Computational and Theoretical Fluid Dynamics Div., National Aeronautical Lab., PD CF 9119, Bangalore, India, Aug. 1991.

[6]Srivastava, A., "Quadrilateral Vortex Panel Code—Numerical Testing," Computational and Theoretical Fluid Dynamics Div., National Aeronautical Lab., PD CF 9123, Bangalore, India, Nov. 1991.

[7]Mook, D. T., and Dong, B., "Numerical Simulations of Wakes and Blade-Vortex Interaction," *Journal of Fluids Engineering*, Vol. 116, No. 1, 1994.